



# Clustering of advected passive sliders on a fluctuating surface

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**Abstract**— We study the clustering properties of advected, non-interacting, passive scalar particles in a Burgers fluid with noise, a problem which maps to that of passive sliding particles moving under gravity on a surface evolving through the Kardar-Parisi-Zhang equation. Numerical simulations show that both the density-density correlation function and the single-site mass distribution scale with system size. The scaling functions diverge at small argument, indicating strong clustering of particles. We analytically evaluate the scaling functions for the two-point correlation and mass distribution of noninteracting particles in thermal equilibrium in a random landscape, and find that the results are remarkably similar to those for nonequilibrium advection.

**Keywords**— Passive scalar, clustering, advection, Burgers equation, KPZ equation

## I. INTRODUCTION

The advection of a passive scalar in a turbulent fluid is an interesting problem which has, in the recent past, attracted considerable attention [1] [2] [3]. By a passive scalar we mean a scalar field (e.g. temperature or density of dye) that is advected by the fluid flow but which has no back effect on the flow. In this paper we will consider the problem of a passive particle density in a compressible fluid described by the Burgers equation in the presence of white noise, studied recently in [4]. While an incompressible density field would tend to disperse and ramify when carried by an incompressible flow, the reverse can happen if the fluid is compressible, and particle clustering (rather than dispersal) can result. The characterization of this clustering in the one-dimensional case is the principal objective of this paper.

The restriction to flows described by the Burgers equation [5] [6] allows the problem to be mapped to another interesting problem, namely that of passive, sliding particles (called sliders) moving under gravity along a growing surface which is itself fluctuating in time. Previous studies of the related problem of hard-core particles sliding on a fluctuating surface have shown the occurrence of a new and interesting state, with large-scale clustering of particles [7] [8]. Now in the problem under consideration here, there is no hard-core interaction between sliders, so we may expect even stronger clustering properties. As discussed below, this expectation is borne out in our numerical simulations of the system. There is a clear signature of clustering in the behaviour of the density-density correlation function  $C(r)$ , which has a scaling form with argument  $r/L$  where  $L$  is the system size. While in the fluctuation-dominated phase ordering (FDPO) state discussed in [7] and [8], the scaling function has a *cusp* singularity at small argument, we find that it has a *divergence* in the problem under study here. Interestingly, we find that this strongly nonequilibrium system has the same scaling properties as a system of noninteracting particles in thermal equilibrium in a random, static landscape. The latter problem is

solved analytically, and the results are found to be similar to the numerical results for nonequilibrium advection.

## II. MODEL

We first discuss the continuum equations describing the passive scalar problem with Burgers flow. We then describe a one dimensional lattice model which is expected to have the same scaling properties.

The velocity field  $\vec{v}$  of a randomly stirred Burgers fluid is described by the equation

$$\frac{\partial \vec{v}}{\partial t} + \lambda(\vec{v} \cdot \nabla)\vec{v} = \nu \nabla^2 \vec{v} + \nabla \zeta_h(\vec{x}, t) \quad (1)$$

Here  $\nu$  is the viscosity, while the random stirring is caused by Gaussian white noise  $\zeta_h$  satisfying  $\langle \zeta_h(\vec{x}, t) \zeta_h(\vec{x}', t') \rangle = 2D_h \delta^d(\vec{x} - \vec{x}') \delta(t - t')$ .

A passive scalar particle moving with the flow follows the equation

$$\frac{d\vec{x}}{dt} = a\vec{v} + \zeta_x(t) \quad (2)$$

where the white noise  $\zeta_x(t)$  represents the randomising effect of temperature, and satisfies  $\langle \zeta_x(t) \zeta_x(t') \rangle = 2\kappa \delta(t - t')$ . The parameter  $a$  governs the coupling of the particle to the flow. If the flow is vortex-free, we may write  $\vec{v} = -\nabla h$ , where the field  $h$  satisfies the equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \zeta_h(\vec{x}, t) \quad (3)$$

This is the Kardar-Parisi-Zhang (KPZ) [9] equation for surface evolution with  $h(\vec{x}, t)$  describing the height of a growing surface at position  $\vec{x}$  at time  $t$ .

Rather than analysing the coupled Eqs. (2) and (3) directly, we will study a lattice model which is expected to have similar behaviour at large length and time scales. The model consists of a flexible one-dimensional lattice in which particles reside on sites, while the links or bonds between successive lattice sites are also dynamical variables denoting local slopes. Each link takes values  $+1$  (upward slope  $\rightarrow /$ ) and  $-1$  (downward slope  $\rightarrow \backslash$ ). We evolve the surface by choosing a site at random, and if it is on a local hill ( $\rightarrow /$ ), the local hill is changed to a local valley ( $\rightarrow \backslash$ ). These moves of the links define the well known single-step model, whose large distance, long time properties are known to be identical to those of the KPZ equation. The particles (any number of which can be present at a given site) are moved one at a time. We choose a particle at random and move it one step downward with probability  $(1 + K)/2$  or upward with probability  $(1 - K)/2$ . The parameter  $K$ , which ranges from 1 (particles totally following the surface slope) to 0 (particles moving independently of the surface) mimic the ratio  $a/\lambda$  in Eqs. (2) and (3). In our simulations, we update the surface and particles at independent sites, reflecting the independence

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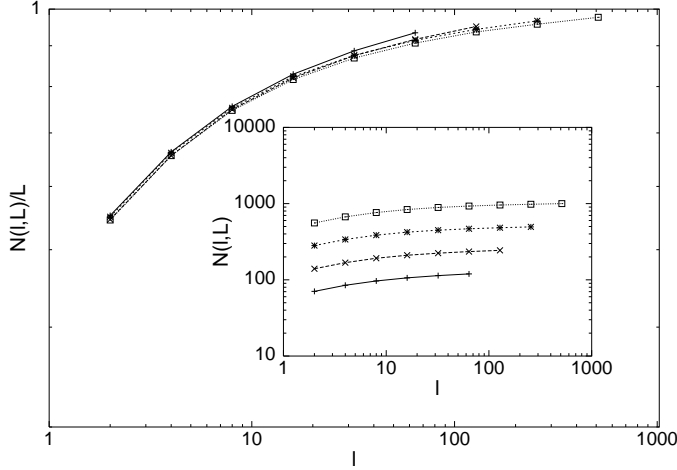


Fig. 1

THE NUMBER  $N(l, L)$  OF PARTICLES WHICH LIE IN THE SAME BIN OF SIZE  $l$  AS A RANDOMLY CHOSEN PARTICLE IS SHOWN IN THE INSET FOR VARIOUS SYSTEM SIZES  $L$ . THE VARIOUS SIZES ARE 128, 256, 512, 1024 AS WE MOVE UP. THE MAIN FIGURE SHOWS THE SCALING OF  $N(l, L)$  WITH  $l$

of the noises  $\zeta_h(\vec{x}, t)$  and  $\zeta_x(t)$ . This is in contrast to Kardar and Drossel's update rule [4] where only particles residing at a site affected by the surface evolution are moved.

### III. NUMERICAL RESULTS

As particles move preferentially downwards, it is evident that they tend to move towards valleys, and this valley-seeking tendency promotes clustering. The question is how one can quantify its nature and extent. The numerical study of the two-point correlation function and single-site particle distribution described in this section show that a scaling description affords a simple and compact description of clustering properties.

We consider the steady state of a system of  $M$  particles in a system with  $L$  sites. In our numerical study, we used  $L$  in the range 128 to 1024, and took  $M = L$ . The successive update times of the particle and the surface moves were taken to be equal to each other. To start with, we monitored the RMS displacement  $x_{RMS}$  of a given tagged particle, and verified that  $x_{RMS} \sim t^{1/z}$  where  $z = 3/2$  is the KPZ value of the dynamic exponent. This is expected on the basis that the particle is likely to be in the largest fresh valley that has formed within time  $t$  in the vicinity of the particle, as such valleys have a spatial size of order  $t^{1/z}$ . Quite direct evidence of particle clustering is obtained by choosing a particle at random and monitoring the mean number of particles within a distance  $l$  of it. The results are shown in Fig. 1. Saturation of the curves as  $l$  increases is a clear indication of particle clustering. These results are essentially identical to those of Drossel and Kardar [4] for their slightly different model.

Our results for the two-point (unconnected) density-density correlation function  $G(r, L) = \langle m_i m_{i+r} \rangle$  for different  $L$  are shown in Fig. 2, where  $m_i$  is the number of particles at site  $i$ . There is strong evidence that the scaling form

$$G(r, L) \sim L^{-\theta} Y(r/L) \quad (4)$$

is valid with  $\theta = \frac{1}{2}$ , and that the scaling function  $Y(y)$  has a power law divergence  $Y \sim y^{-\phi}$  as  $y \rightarrow 0$ . The value of  $\phi$  is close to 1.5, which is the value obtained for particles which have reached equilibrium on a random surface, as discussed in section IV below.

The divergence of the scaling function at small argument is the result of strong clustering. It is an outcome of the unrestricted occupancy of a site by noninteracting particles, and quite different from the milder cusp singularity found in the FDPO state with hard-core exclusion [8]. As shown in Section IV, the form of the scaling function can be found analytically for the ostensibly different problem of noninteracting particles in thermal equilibrium on a random landscape. As a matter of fact, the analytic forms describe quite well the data in Fig. 2 for the strongly nonequilibrium system under study. This is discussed further in Section V.

Another interesting quantity, which however does not carry any spatial information, is the probability  $P(m, L)$  that any site has occupancy  $m$ . As is evident from Fig. 3,  $P(m, L)$  follows the scaling form

$$P(m, L) \sim \frac{1}{L^2} f\left(\frac{m}{L}\right). \quad (5)$$

At first sight, the scaling function seems to follow a power law  $f(y) \sim y^{-\tau}$  with  $\tau$  close to 1, but guided by the analytic theory to be discussed in the next section, we find that the  $y \rightarrow 0$  behaviour is well described by the form  $f(y) \sim y^{-1} \ln(b/y)$  with  $b \simeq 5$ . On integrating over  $m$ , we find  $1 - P(0, L) \sim \frac{1}{L} (\ln L)^2$ , implying that the number of occupied sites increases with size as  $(\ln L)^2$ . This indicates that the particles are distributed within a relatively compact region in space. Finally, we note that Eq.(5) leads to  $\langle m^2 \rangle \equiv G(0, L) \sim L$ . Direct simulation results for  $G(0, L)$  verify the linear dependence on  $L$ , lending further support to the scaling form (Eq.(5))

### IV. ANALYTICAL RESULTS (EQUILIBRIUM)

Since the particles are noninteracting in our model, the number density  $\rho(r, t) = m(r, t)/L$  of particles is identical to the probability that a single particle, moving in the fluctuating landscape, will be at a position  $0 \leq r \leq L$  at time  $t$ . Thus it suffices to consider one single particle moving in the fluctuating landscape. We consider the limiting case when changes in the surface occur over a time scale much larger than the time scale of fluctuations of the particle. In this limit, for any given surface configuration  $h(r)$ , the Langevin equation (Eq.(2)) describes the evolution of a particle in a potential  $h(r)$ . Before the surface configuration  $h(r)$  can change, the particle will relax to its equilibrium Gibbs state, where the probability of finding the particle at position  $r$  is given by  $\rho(r) = \exp[-\beta h(r)]/Z$  with the partition function  $Z = \int_0^L \exp[-\beta h(r')] dr'$ , where  $\beta = 1/k_B T$ . Clearly  $\rho(r)$  is a random variable which changes from one configuration of  $h(r)$  to another. We assume that the surface itself has reached the stationary state of Eq. (3). It is well known that for the 1-d KPZ equation, the stationary state is described by the following measure,  $\text{Prob}[\{h(r)\}] \propto \exp[-\frac{1}{2} \int h^2(r') dr']$ . Thus, any stationary configuration can be thought of as the trace of a random walker in space evolving via the equation,  $dh(r)/dr = \xi(r)$  where the white noise  $\xi(r)$  has zero mean and

is delta correlated,  $\langle \xi(r)\xi(r') \rangle = \delta(r - r')$ . Thus our problem then reduces precisely to the celebrated Sinai model [11] where a particle moves in a random potential which itself is a random walk in space. There is a slight difference however. The periodic boundary condition on the surface,  $h(r) = h(r + L)$ , indicates that we are considering a random potential which is pinned at its two ends, i.e. a Brownian bridge, rather than a free walk.

Given this stationary measure of the surface configurations, we then want to evaluate the correlation function  $\langle \rho(r_0)\rho(r + r_0) \rangle = G(r, L)/L^2$  (evidently independent of  $r_0$  due to translational invariance) where the angular brackets denote the average over the surface configurations sampled from the stationary measure mentioned above. Thus we have

$$L^{-2}G(r, L) = \left\langle \left[ \frac{e^{-\beta[h(r_0)+h(r_0+r)]}}{Z^2} \right] \right\rangle. \quad (6)$$

Fortunately, the object on the right hand side of Eq.(6) was evaluated exactly by Comtet and Texier [10] in the completely different context of one dimensional disordered supersymmetric quantum mechanics, where the right hand side of Eq. (6) is simply the correlation function in the ground state wave function. They actually evaluated the  $n$ -point correlation function. Adapting their results to our case with  $n = 2$ , we find the following expression

$$G(r, L) = \sqrt{2\pi} \frac{(\beta^2 L)^{5/2}}{256} \int_0^\infty \int_0^\infty dk_1 dk_2 k_1 k_2 (k_1^2 - k_2^2)^2 \times \frac{\sinh(\pi k_1) \sinh(\pi k_2)}{[\cosh(\pi k_1) - \cosh(\pi k_2)]^2} \exp \left[ -\frac{\beta^2}{8} (k_1^2(L-r) + k_2^2 r) \right]. \quad (7)$$

Note that the expression for  $G(r, L)$  has the expected symmetry,  $G(r, L) = G(L - r, L)$ . It is easy to evaluate  $G(0, L)$  from Eq. (8) for large  $L$ . We find,  $G(0, L) \approx \beta^2 L/12$  for large  $L$ . On the other hand, in the scaling limit,  $r \rightarrow \infty$ ,  $L \rightarrow \infty$  but keeping the ratio  $y = r/L$  fixed, one finds from Eq. (7) that for  $0 < y < 1$ ,  $G(r, L) \sim L^{-1/2} Y(r/L)$  where the scaling function is given by

$$Y(y) = \frac{1}{\beta\sqrt{2\pi}} [y(1-y)]^{-3/2}. \quad (8)$$

Note that the point  $r = 0$  is not part of the scaling function. In fact, the power law divergence of the scaling function as  $y \rightarrow 0$ ,  $Y(y) \sim y^{-\phi}$  with  $\phi = 3/2$ , is necessary in order that  $G(r \rightarrow 0, L) \sim L$  for large  $L$ .

This formalism can also be used to calculate the equilibrium probability density  $P(\rho, L)$ . Details of the calculation will be given elsewhere. Our results indicate that  $P(\rho, L)$  which can be written as the sum of two parts :

$$P(\rho, L) \approx \left[ 1 - \frac{\ln^2(L)}{\beta^2 L} \right] \delta(\rho) + \frac{4}{\beta^4 L} G \left[ \frac{2\rho}{\beta^2} \right] \theta \left( \rho - \frac{c}{L} \right), \quad (9)$$

The first part refers to vacant stretches, and to the fact that the number of occupied sites occupies a vanishing fraction  $\sim (\ln L)^2/L$  of the system. The scaling function  $G(y)$  in the second part is given by

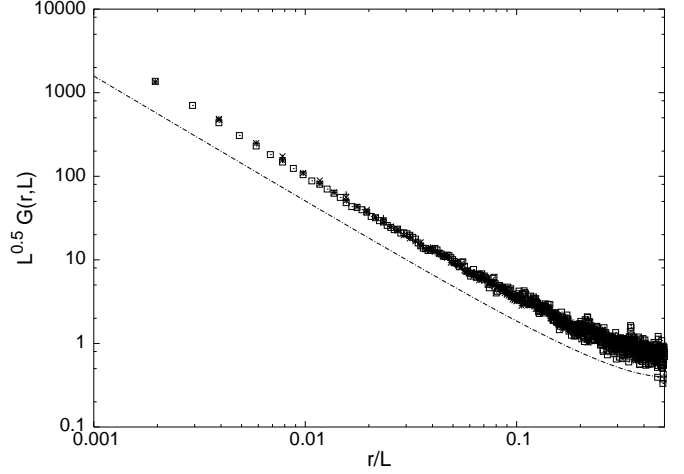


Fig. 2

SCALING OF THE DENSITY-DENSITY CORRELATION FUNCTION  $G(r, L)$  WITH SYSTEM SIZE  $L$ . THE DASHED CURVE IS A PLOT OF THE SCALING FUNCTION  $Y$  (EQ.(8)) SHIFTED DOWNWARDS FOR CLARITY.

$$G(y) = \frac{e^{-y}}{y} K_0(y). \quad (10)$$

where  $K_0(y)$  is the modified Bessel function which has the asymptotic behaviour  $[-\ln(y/2) - 0.5772\dots]$  as  $y \rightarrow 0$ . The theta function incorporates a lower cutoff on the validity of the scaling form.

The equilibrium-based results obtained in this section appear to describe rather well the behaviour of the nonequilibrium system studied numerically in section III. For instance Eq.(8) describes the behaviour (Eq.(4)) of the two-point correlation function (Fig. 2), with  $\beta \simeq 4$  giving good agreement. The data for the mass distribution (Fig. 3) is quite well described by Eq.(10), but with a different value of  $\beta$  ( $\simeq 2.3$ ). An intermediate value of  $\beta$  ( $\simeq 2.7$ ) gives tolerable, but not very satisfactory, agreement with both sets of data (Figs. 2 and 3).

## V. DISCUSSION

As we have seen in the previous two sections, a number of the properties of the nonequilibrium system of particles sliding in a fluctuating landscape are strikingly similar to those of an equilibrium system of particles settling in a quenched random landscape. This raises two questions: First, what is the source of this similarity in behaviour? And second, how robust is it? We address these below.

In thermal equilibrium, in a given configuration of the landscape, the Boltzmann factor guarantees that most particles are to be found within a height  $T$  of the global minimum, where  $T$  is the temperature. In the nonequilibrium situation, by contrast, examination of several typical configurations shows that particles cluster near valley bottoms, but these valleys are often not the globally lowest one. Why then are properties similar when averaged over configurations? A partial answer lies in

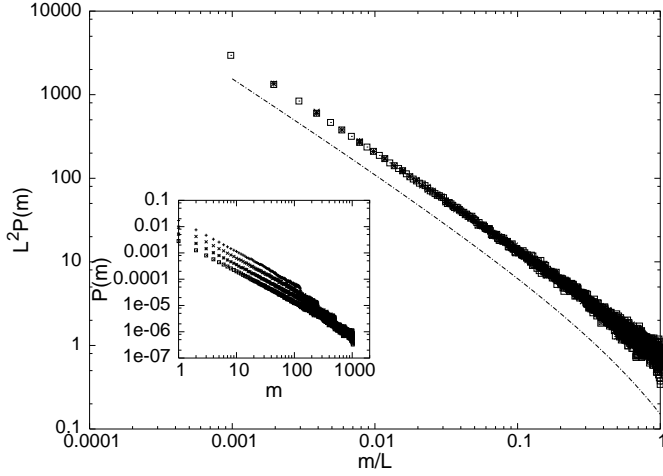


Fig. 3

THE PROBABILITY  $P(m, L)$  OF FINDING A SITE CONTAINING  $m$  PARTICLES IS SHOWN IN THE INSET FOR VARIOUS SYSTEM SIZES  $L$ . THE MAIN FIGURE SHOWS THE SCALING OF  $P(m, L)$  WITH  $L$ ; THE DASHED CURVE SHOWS THE SCALING FUNCTION  $G$  (EQ. (10)) SHIFTED DOWNWARDS FOR CLARITY.

the fact that terrain around *any* reasonably deep valley is statistically similar. This feature of the ‘random walk’ landscape under consideration can plausibly lead to particles being distributed in similar fashions in the two cases.

Evidently, this happens despite the fact that the character of the noise which leads to particle re-settling is quite different in the two cases: in the equilibrium problem, particles are acted upon by thermal noise, while in the nonequilibrium case, it is surface fluctuations which drive re-settling. Thus, despite the similarities in the landscape, it is not clear to what extent one should expect an equivalence of results, and indeed, as the comparison in Section IV shows, the equivalence for different properties involves equilibrium systems at different temperatures. It would also be interesting to see how robust these results are with respect to variations of the parameter  $\omega$ , which is the ratio of the times  $\tau_s$  between successive updates of the surface, and  $\tau_p$  between successive updates of the particles. In the studies reported in this paper,  $\omega$  was held constant at 1. Preliminary investigations suggest that variations of  $\omega$  can induce interesting features specific to the nonequilibrium problem.

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## REFERENCES

- [1] R.H. Kraichnan, “Anomalous scaling of a randomly advected passive scalar”, *Phys. Rev. Lett.* vol. 85, pp. 1016-1019 (1994)
- [2] B. I. Shraiman and E. D. Siggia, “Scalar turbulence”, *Nature* vol. 405, pp. 639-646 (2000)
- [3] G. Falkovich, K. Gawedzki, and M. Vergassola, “Particles and fields in fluid turbulence”, *Rev. Mod. Phys.* vol. 73, pp. 913-975 (2001)
- [4] B. Drossel and M. Kardar, “Passive sliders on growing surfaces and advection in Burgers flows”, *Phys. Rev. B* vol. 66, pp. 195414-1-4 (2002)
- [5] J. M. Burgers, *The non-linear diffusion equation: asymptotic solutions and statistical problems*. Kluwer Academic Publishers, 2002

- [6] F. Hayot and C. Jayaprakash, “Aspects of the stochastic Burgers equation and their connection with turbulence”, *Int. J. Mod. Phys. B*, vol. 14, pp. 1781-1800 (2002)
- [7] D. Das and M. Barma, “Particles sliding on a fluctuating surface: phase separation and power laws”, *Phys. Rev. Lett.* vol. 85, pp. 1602-1605 (2000)
- [8] D. Das, M. Barma and S. N. Majumdar, “Fluctuation-dominated phase ordering driven by stochastically evolving surfaces: depth models and sliding particles”, *Phys. Rev. E* vol. 64, pp. 046126-1-16 (2001)
- [9] M. Kardar, G. Parisi, Y. Zhang, “Dynamic scaling of growing interfaces”, *Phys. Rev. Lett.* vol. 56, pp. 889-892 (1986)
- [10] A. Comtet and C. Texier, “One-dimensional disordered supersymmetric quantum mechanics: a brief survey” in *Supersymmetry and Integrable Models*, Aratyn, Imbo, Keung, Sukhatme (eds.), Proceedings Chicago IL, 1997, Lecture notes in Physics, Springer, (1998); cond-mat/9707313
- [11] Y.G. Sinai, “The limiting behavior of a one-dimensional random walk in a random medium”, *Theor. Probab. Appl.* vol. 27, pp. 256-268 (1982).
- [12] M. Gopalakrishnan, “Anomalous diffusion of a passive sliding particle on a fluctuating surface”, *cond-mat/0308109*